Demand for Sovereign Safe Assets*

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Abstract

What explain the decline in the real rates despite soaring government debt? What do low rates imply about the riskiness of government debt? I show that government bond becomes a safe asset, when government default significantly reduces the returns to other assets. A negative shift in fundamentals of the safe asset increases its price and the risk premium. When government bond is a safe asset, it can have an upward-sloping demand curve for intermediate levels of borrowing. This happens because more borrowing increases aggregate risk and demand for safety, while government bond remains the relatively safer asset. In such a case, an increase in government borrowing increases the price of bonds, reduces the price of other assets, and increases the risk premium. The existence of an upward-sloping demand curve depends on the reversed hazard rate of the distribution of future tax revenues. Factors that increase the systemic risk of government debt, including higher fraction of government debt to other investors' assets, higher riskiness of the payoffs to other assets and investors' indebtedness, make an upward-sloping demand curve more likely. These findings also suggest more caution in interpreting the currently low real rates, as price of a safe asset and its default risk can comove positively.

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1 Introduction

Despite a significant surge in government borrowing after the Financial Crisis of 2008 and during the ongoing COVID-19 pandemic, real interest rates on government bonds have been declining in many economies (Figure 1). It has been argued that low real interest rates might imply that more government borrowing have low fiscal costs (Blanchard (2019)). Others have cautioned that even though the current rates are low, more borrowing can make government subject to bad self-fulfilling equilibria (Farhi and Maggiori (2018), Cochrane (2021)). What can explain the decline in the real rates on government debt? What do these low rates imply about the riskiness of government debt?

I show that when government debt is systemic, a deterioration in the distribution of future tax revenues or an increase in government debt can reduce the real rate on government bonds, while increasing the risk of default. Government defaults often lead to severe contractions in output (Mendoza and Yue (2012), Trebesch and Zabel (2017)). These contractions can be even more severe when government defaults are accompanied by banking crises (Gennaioli et al. (2014)). Government default can also be detrimental to the economy, when a significant fraction of government debt is held by domestic shadow banks. For example, 12% of U.S. government debt is held by U.S. mutual funds. Moreover, when significant quantities of government debt is held externally, a government default can adversely affect the balance sheets of foreign holders of the debt. For example, significant fractions of U.S. and German debt are held by foreign investors.[1][2]

In this paper, I study the demand for systemic government debt. I show that government bond becomes a safe asset if government default significantly reduces the return to other existing assets ,i.e., entails systemic risk. When government bond is a safe asset, a negative shift in or an increase in riskiness of the distribution of future tax revenues *increase* the bond price and the default risk at the same time. Such changes in future economic prospects also decrease the price of other assets, and increase the risk premium. Moreover, under such conditions, the government might face an upward-sloping demand curve for intermediate levels of borrowing: more issuance *increases* both the bond price and the default risk. Under an upward-sloping demand curve, more borrowing by the government increases the risk premium. I also show the upward-sloping segment of the demand for government debt always ends in an inflection point: for high enough level of debt, further supply decreases the price.

Shortage of safe assets has been argued to be a major cause of the decline in the real rates (Caballero et al. (2017), Del Negroa et al. (2019)).[3] My theory suggests that the significant increase

[1]See https://www.cbo.gov/publication/56309, for the composition of U.S. debt holders.

[2]In the case of U.S. government debt, a default can be even more costly due to the especial role of dollar assets in the global trade and finance. See Gourinchas (2019).

[3]There are more than a few explanations for the secular decline in the real interest rates. I consider my model as

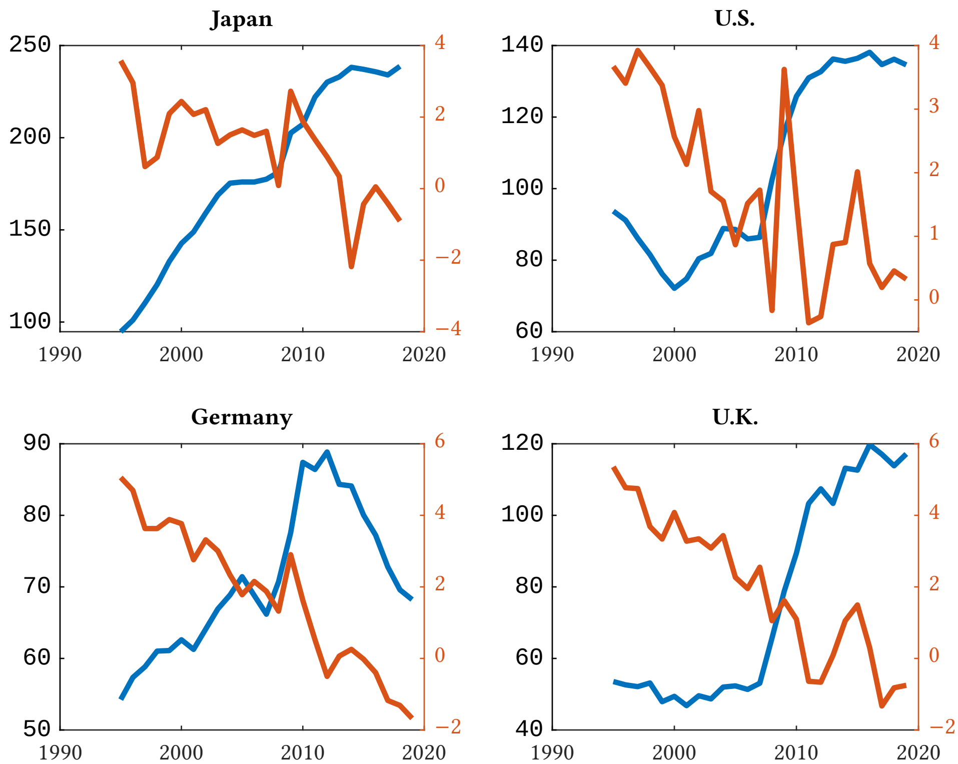

Figure 1: Government debt to GDP in blue (left axis), and long-term (10 years) real rates in red (right axis). Both are in percents. Data is from OECD and the author's calculations.

in the supply of government debt and its systemic risks especially after the Financial Crisis of 2007-2008, may have exacerbated this "shortage" and contributed to even lower levels for real rates. In other words, supply of systemic safe assets by governments may have increased the demand for the same assets.

A testable prediction of the model is that, all else equal, higher systemic risk of government debt makes an upward-sloping demand curve more likely. Factors that increase systemic risk include higher fraction of government debt to other investors' assets, higher riskiness of the payoffs to other assets, and higher investors' indebtedness. The holdings of own government debt by banks are significant in both advanced economies and emerging markets. The evidence also suggests that banks in advanced economies and emerging markets have significantly increased their holdings of own sovereign debt after 2007, thereby increasing the systemic risk of government debt (Dell'Ariccia et al. (2018), Feyen and Zuccardi (2019), Arslanalp and Tsuda (2014)). My findings also imply that when government bond is systemic, the price of bonds, i.e., inverse of real borrowing rates, and the risk of default might comove positively. Under such conditions,

a complementary to the explanation based on safe asset shortage. For more theories on the decline of the real rates, see Del Negroa et al. (2019), and references therein.

bond price can be a misleading indicator of the default risk. Moreover, the existence of inflection points suggests that the real rates will start rising for high enough levels of government debt.

A brief description of the model is as follows. I consider an economy with two dates and two assets: government bond and a real asset or "project" that is in fixed supply. There are three agents, a government, a continuum of investors and a representative household. The government has to raise resources by issuing bonds to pay for an exogenous amount of spending. Investors are the only agents that have resources to purchase government bonds at the initial date. One can think of investors in my model as banks or shadow banks. Household's future income is stochastic and can be taxed, up to a limit, by the government to repay its debts to investors. The government pays its debts in full as long as there are enough tax revenues, and otherwise defaults. In the event of a default, investors are paid an exogenous fraction $0 \leq z < 1$ of the promised debt payments. Besides government bonds, investors hold a real asset or "project" each, at the beginning of the initial period. The investor needs to pay a fixed and exogenous maintenance cost for the project in the second period, in order to guarantee a high return on the project. This maintenance cost can capture any liquidity needs in a stylized way. Examples include rolling over maturing debt or financing working capital. If the maintenance cost is not paid, the project would have a low return with an exogenous probability $0 < p \leq 1$, which is independent across the continuum of projects. The only resources the investor can use in the second period to cover the project's maintenance cost is the government repayments of its debt. Hence, when government defaults, investors holding the bonds may not be able to pay for the project maintenance costs. This implies that government default affects the return on projects. The probability p is a measure of correlation between the risks to the government bonds and the project. A higher p implies a higher likelihood of low returns to both assets happening at the same time, and more aggregate risk.

The intuition behind an upward-sloping demand curve is as follows. Higher supply of government bonds affects the price through two channels. First, even absent default, a higher level of borrowing by the government increases the consumption of investors and reduces the expected marginal utility of an extra unit of investment in the bond. This unambiguously reduces the price of bond. I call this the *diminishing expected marginal utilities* effect. At the same time, a higher level of borrowing increases the probability of default. When project's expected payoff depends on the incident of default, this increases the demand for insurance by investors and tends to increase the price of bond. The reason is that default reduces the return to the project significantly so that government bond still remains the safest asset. I call this the *change in aggregate risk* effect. A positive sum of *diminishing expected marginal utilities* and *change in aggregate risk* effects implies an upward-sloping demand curve.

I demonstrate that, in general, the net impact of *diminishing expected marginal utilities* and *change in aggregate risk* effects depends on the reversed hazard rate of the future tax revenues distribution. I show that when the density function of future tax revenues is convex on its left tail, and the return to other assets in the no-default state is high enough, investors have an upward-sloping demand for some values of bond holding. Convexity of the density function on the left tail provides a sufficient, and not necessary, condition for the existence of a (partially) upward-sloping demand curve. Nonetheless, this condition holds for many distributions including normal, log normal, and Weibull distributions.

There are two necessary conditions on the parameters of the model for the government bonds to have an (partially) upward-sloping demand curve. First, given the high and low values for the project return, the default value or the recovery rate of the bond should not be too low (and strictly above zero). The reason is that when the default value is too low, government bond becomes riskier than the project. In this case, higher risk of default lowers the (negative) risk premium on the bond and reduces the price of bond. Second, the probability p has to be higher than a threshold, which is a function of the return profile of the project. The reason is that a high value of p is needed so that default can create significant aggregate risk for the investors.

I introduce a market for shares in the projects held by investors. I show that when the demand for government bond is upward-sloping, an increase in borrowing increases the relative price of bonds to shares or what I call the risk premium. The reason is that government bond is the safe asset in this case. An increase in the aggregate risk increases the demand for insurance which raises the relative demand for government bonds.

In my benchmark model, I use maintenance cost as the micro foundation to produce the main results. There are other mechanisms that can make government bonds a safe asset and can potentially lead to an upward-sloping demand curve. I consider one such alternative mechanism. I study a model similar to the benchmark with one important change. Instead of the project, investors hold old government bonds from previous periods, which has been partly financed by previously issued liabilities. One can think of these previously issued liabilities as any type of funding for the pre-existing government debt, such as deposits, repo, and commercial paper. I assume that both the old bonds and old liabilities mature in the second period. The key assumption is that a default on newly issued bonds leads to a default on the old bonds. Under these conditions, when the sum of old and new bonds are large enough, the demand curve can become upward-sloping. This implies that for low levels of new bond issuance, a high level old bonds makes an upward-sloping demand curve more likely. Moreover, I show that under certain conditions, an increase in the investor's pre-existing debt increases the slope of the demand curve for the newly issued bonds. The reason is that a higher level of old liabilities makes the return to the old assets and liabilities riskier for any given level of default probability.

The rest of the paper is organized as follows. Section 2 discusses the related literature. In

Section 3, I present the benchmark model. I discuss the balance sheet effect in Section 4 as another mechanism to reach the same results as in the benchmark model. Finally, Section 5 concludes.

2 Related Literature

Liu (2019) and Cochrane (2020) find that higher debt to GDP ratios are associated with lower real interest rates. These findings are consistent with an upward-sloping demand curve for the U.S. debt. Liu (2019) finds that higher debt to GDP ratios are associated with higher measures of fiscal uncertainty, which can explain the the effect of higher debt on risk premia and real rates. My model is complementary to the mechanism in Liu (2019).

Krishnamurthy and Vissing-Jorgensen (2012) find that a higher U.S. debt to GDP reduces both liquidity and safety premium of Treasury debt. In contrast, Liu (2019) and Liu et al. (2020) find that an increase in the U.S. debt to GDP ratio predicts higher excess stock returns and is correlated with higher credit risk premia. My model is consistent with the latter findings.

He et al. (2019) present a model of safe asset determination in a two country setup. They find that a larger bond issuance and better relative fundamentals can make a country's bond safe. He et al. (2019) also find that a deterioration in the world-level fundamental can increase the price of the safe country's bond, while reducing the price of the other, i.e., flight to safty. In my model, government bonds can become safe asset when government default dampens the return to the other assets. Moreover, *even* a deterioration in fundamentals of the issuer of the safe asset increases the price of bonds and increases the risk premia.

In Farhi and Maggiori (2018), the supply of the safe (or reserve) asset can make the hegemon country susceptible to self-fulfilling confidence-crisis. Although there is no confidence-crisis in my model, I show the nature of a safe asset such as U.S. Treasuries can weaken or even reverse the impact of higher risks on the bond price, and risk premia.

Several recent papers, including Chernov et al. (2020) and Dittmar et al. (2019), study the U.S. sovereign default risk. Chernov et al. (2020) show that the U.S. CDS premiums, which have been elevated since the Financial Crisis of 2008, reflect the endogenous risk-adjusted probabilities of fiscal default. Dittmar et al. (2019) examine the relative pricing of nominal Treasury bonds and Treasury inflation-protected securities (TIPS) in the presence of United States default risk. They show that most of the relative mispricing after the crisis is due to default risk.

Gennaioli et al. (2014) find that domestic government debt held by banks is, on average, 11.8% of their total assets. They also document that 67% of government defaults are accompanied by banking crises in their sample. This suggests that government bond is likely to be systemic in many countries, and shows the plausibility of the mechanism modeled in this paper.

3 Model

There are two periods $t = 0, 1$. The economy consists of a government, a representative household, and a measure one of ex-ante identical investors. The model economy can capture both a closed economy and the global economy. The main difference in these two interpretations is that investors are domestic in one and foreign in the other.

The household has no income at $t = 0$ but receives a stochastic endowment y_1 at $t = 1$. I assume that only a maximum fraction $0 < \tau \leq 1$ of y_1 can be taxed by the government, and that τy_1 is distributed according a cumulative distribution function F. Suppose that the household consumes only in $t = 1$. To avoid a large utility cost to the household, the government has to finance a deficit equal to g_0 units of $t = 0$ consumption goods. To do that, the government can sell b_1 units of bonds to the investors, at time $t = 0$, each unit of which pays off one unit of consumption goods at $t = 1$. Let q_0 denote the price of this bond in units of time $t = 0$ consumption goods. The government uses the tax receipts τy_1 to pay back its debt. Since y_1 is stochastic, for high enough b_1, the government may not be able to pay its debt obligations in full and defaults. If the government defaults on its debt, I assume that the creditors can recover zb_1, where $0 \leq z < 1$. I assume that the government debt pays off at the beginning of $t = 1$. Each investor has an endowment y_0 at $t = 0$, and the following preferences:

$$\boldsymbol{U}^* = c_0 + \mathbb{E}_0\Big[\ln(c_1)\Big].$$

c_0 and c_1 are consumption at the current and future dates. Note that there is no time discounting for simplicity. Each investor also holds a project, i.e., real asset, at the initial date that pays off at the end of $t = 1$. Before its completion, the investor needs to spend $\delta > 0$ units of time $t = 1$ consumption goods at the beginning of $t = 1$ in order to maintain the project. If the full maintenance cost δ is paid, the project yields a payoff of a_H. Otherwise, if δ is not paid, the project's payoff decreases to $0 < a_L < a_H$ with probability $p \in [0, 1]$, or remains a_H with probability $1 - p$. The maintenance cost in my model is similar to the liquidity shock in Holmström and Tirole (1998). The difference is that unlike Holmström and Tirole (1998), the return in the event of not covering the liquidity needs can be stochastic while the size of the liquidity shock is not. I assume that the realization of project's payoff across different projects are i.i.d across investors, and that this idiosyncratic risk can not be insured. The following assumption ensures that it is always optimal for the investor to pay the maintenance cost if she can:

Assumption 1. $p \geq \frac{\delta}{a_H - a_L}$.

One can think of investors as banks, shadow banks, or any other entity that might entail significant negative spillovers to the real sector of the economy under financial stress. Gennaioli et al. (2014) find that domestic government debt held by banks is, on average, 11.8% of their total assets. They also document that 67% of government defaults are accompanied by banking crises in their sample. This suggests that it is plausible to think of investors in my model as banks or shadow banks.

Project in this model can capture real investments by banks and shadow banks. Banks and shadow banks often need to provide liquidity to their borrowers in the corporate sector to pay off short term debt or to use as working capital. The maintenance cost in my model, is an stylized way to capture this type of liquidity provision. In this interpretation, the probability p depends on factors such as the leverage of the borrowers in the household or non-financial corporate sector. The reason is that liquidity problems are more likely to turn into solvency problems, when borrowers have high leverage.

Note that Assumption 1 implies that $a_H - \delta \geq a_L$. The timing is as follows. At $t = 0$, investor decides how many government bonds to buy. At $t = 1$, first, the government pays off its debt or defaults. Next, investor pays the maintenance cost of the project, using what she receives from the government. Finally, the project pays off and investor consumes whatever consumption goods are left after paying for the maintenance.

Discussion of the Assumptions The assumption that the household does not have any endowments at $t = 0$, is to simplify the analysis. Similar results can be obtained, if the household can also purchase government bonds, as long as it is the investors who price the assets at the margin.

The quasi-linear preferences for investors also helps to further simplify the characterization of the demand for government borrowing by eliminating the income effects. For other preferences, as long as the income effect is not too large, the results obtained in my model should hold. Moreover, if the marginal investor already holds government bonds before purchasing any new bonds at the initial date, the income effect of a higher bond price can increase, rather than decrease, the investor's demand.

Note that, default in my model can capture the possibility of either a fiscal default or higher than expected inflation. Moreover, I assume that government does not default strategically: it pays its debt obligations as long as its feasible to do so. But, one can write a similar model with strategic default that yields the same qualitative predictions. To do that, I can introduce a stochastic fixed cost of default. This cost reflects reputational cost or costs associated with losing access to world market. It can depend on macroeconomic conditions, which make the cost stochastic.

Finally, there are other micro-foundations that can generate the same demand function for government bonds. I study one alternative in later sections, which uses the balance sheet effect of government bond holdings.

3.1 Equilibrium

I start with the investor's optimal consumption-saving problem. When $b_1 < \delta$, the investor cannot pay the maintenance costs, whether or not government defaults. In contrast, when $\frac{\delta}{z} \leq b_1$, the investor can always pay for the maintenance costs of the project. In Lemma 1, I provide the resulting demand curve in these two cases.

Next, I focus on the most interesting case, in which the investor can pay for the maintenance cost only when the government is not in default. Note that this happens only if $\delta \leq b_1 < \frac{\delta}{z}$. Within this range, investor solves:

$$\begin{cases} U = \max_{c_0^*, b_1} c_0^* + \Big\{(1 - \pi_D)\ln(a_H - \delta + b_1) + \pi_D\Big[(1-p)\ln(a_H + zb_1) + p\ln(a_L + zb_1)\Big]\Big\}, \\ \\ s.t. \quad c_0^* + q_0 b_1 = y_0^* . \end{cases}$$

π_D is the probability of default by the government, and q_0 is the price of bond, which are taken as given by the investor. Given the quasi-linear preferences, the first order condition (FOC) with respect to c_0^* implies a Lagrange multiplier for the budget constraint equal to one. Using this fact along with the FOC with respect to b_1 give the following equation for the demand for government bonds:

$$q_0 = \frac{1 - \pi_D}{a_H - \delta + b_1} + \pi_D\Big[\frac{(1-p)z}{a_H + zb_1} + \frac{pz}{a_L + zb_1}\Big]. \tag{1}$$

Government defaults only if the tax receipts are less than the debt payments, or $b_1 > \tau y_1$. Therefore, the probability of default is equal to $\pi_D = F(b_1)$. Putting this back into 1, gives the following result:

Lemma 1. *The following characterizes the demand for government bonds by investors:*

$$q_0 = \begin{cases} \Big[\frac{(1-p)}{a_H+b_1} + \frac{p}{a_L+b_1}\Big](1 - F(b_1)) + \Big[\frac{(1-p)z}{a_H+zb_1} + \frac{pz}{a_L+zb_1}\Big]F(b_1) & if \quad b_1 < \delta\,, \\ \\ \frac{1-F(b_1)}{a_H-\delta+b_1} + \Big[\frac{(1-p)z}{a_H+zb_1} + \frac{pz}{a_L+zb_1}\Big]F(b_1) & if \quad \delta \leq b_1 < \frac{\delta}{z}\,, \\ \\ \frac{1-F(b_1)}{a_H-\delta+b_1} + \frac{zF(b_1)}{a_H-\delta+zb_1} & if \quad \frac{\delta}{z} \leq b_1\,. \end{cases}$$

Moreover, $\frac{\partial q_0}{\partial b_1} < 0$ *for all* $b_1 \notin [\delta, \frac{\delta}{z})$.

Whenever $b_1 \notin [\delta, \frac{\delta}{z})$, the demand for government bonds is downward sloping. Also, note that when there is no default, i.e., $z = 1$, investors always pay the maintenance costs for $b_1 \geq \delta$, and do not pay the maintenance costs for $b_1 < \delta$. In this case the demand collapses to $q_0 = \frac{1}{a_H-\delta+b_1}$ and $\frac{(1-p)z}{a_H+zb_1} + \frac{pz}{a_L+zb_1}$, for $b_1 \geq \delta$ and $b_1 < \delta$, respectively. In either case, the demand is strictly decreasing in b_1.

Now, consider the case of $z = 0$. In this case, the second term on the RHS of the demand schedule above is zero. Given that $F(b_1)$ is increasing in b_1, the RHS and hence q_0 is unambiguously decreasing in b_1. This implies that in order to have a upward-sloping demand curve, one needs $z > 0$. More generally, taking the derivative with respect to b_1, one obtains:

$$\frac{\partial q_0}{\partial b_1} = \underbrace{-\Big\{\frac{1-F(b_1)}{(a_H-\delta+b_1)^2} + \Big[\frac{(1-p)z^2}{(a_H+zb_1)^2} + \frac{pz^2}{(a_L+zb_1)^2}\Big]F(b_1)\Big\}}_{\text{A}} + \tag{2}$$

$$\tag{3}$$

$$+ \underbrace{\Big[\frac{(1-p)z}{a_H+zb_1} + \frac{pz}{a_L+zb_1} - \frac{1}{a_H-\delta+b_1}\Big]f(b_1)}_{\text{B}}\ . \tag{4}$$

$f(b_1)$ is the probability density function associated with $F(b_1)$. There are two parts to RHS of the equation above. The first part, denoted by A, is the result of *diminishing expected marginal utility* of consumption: holding more bonds results in more consumption, and a lower marginal utility of purchasing an extra unit of bond. This effect is present even absent any risks, and is unambiguously negative implying a lower price.

The second part of RHS in 2, denoted by B, captures the *change in aggregate risk*. B appears in the equation for the price because an increase in borrowing creates more default risk. This component takes into account the increase in risk that is due to higher probability of default, measured by $f(b_1)$. B also depends on the difference between marginal utilities of an extra unit of bond across the two states of default and no-default. When this difference is negative, purchasing one extra unit of bond creates reduces the insurance provided by the government bond, because the investor gains less utility in default relative to the no-default state. Therefore, in this case an increase in b_1 lowers the the bond price even further. This is the case, for example, when p is too low. The reason is that, given $a_H - \delta < \frac{a_H}{z}$, when p is too low, the marginal utility gained from an extra unit of bond is always lower in the state of default. Moreover, when $a_H - \delta < \frac{a_L}{z}$, a marginal increase in bond holding reduces B, regardless of the value of p.

B becomes positive when p is high and $a_H - \delta > \frac{a_L}{z}$. If the increase in the aggregate risk is large enough, the net effect of A and B can become positive. In such a case, government bonds face an upward-sloping demand curve. In other words, the increase in the riskiness of the safe asset can increase the aggregate risk and the demand for insurance, thereby increasing the demand for the safe asset.

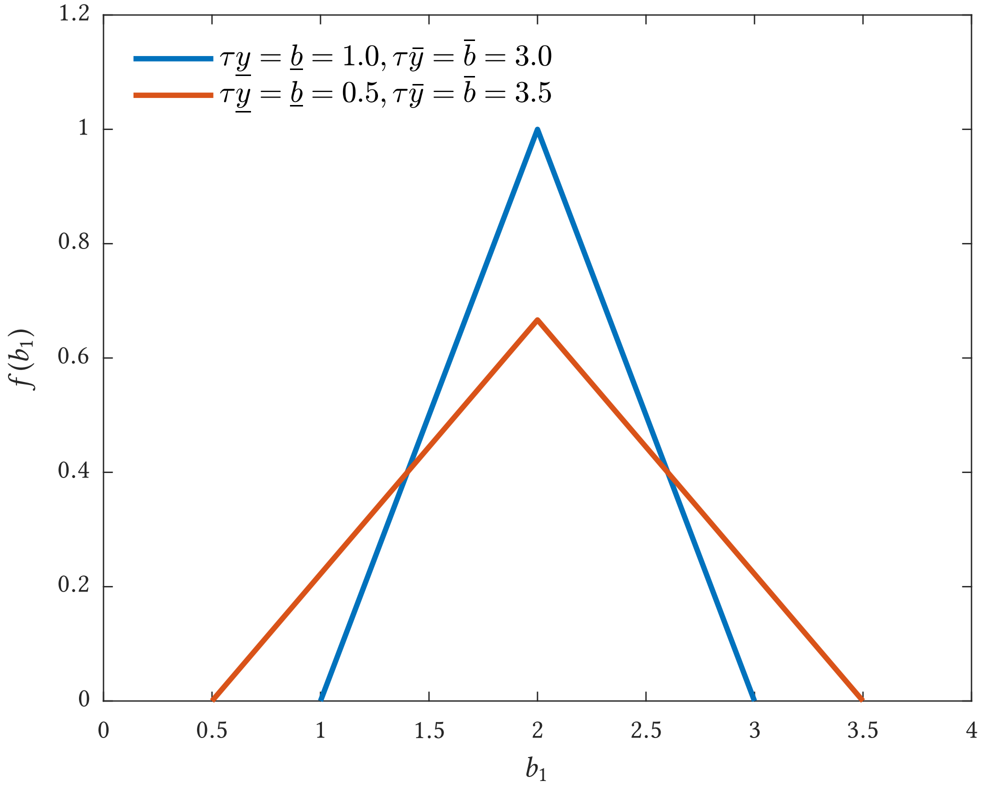

Figure 2: Triangular distributions of future tax revenues. The red density function is a mean preserving spread of the blue density function.

Finally, to close the model, one needs to impose the marketing clearing in the bonds market. The supply curve of government bonds dictates that $g_0 = q_0 b_1$. This defines the supply curve as a decreasing function of b_1. The intersection between the supply and demand gives the equilibrium price and quantity (Figure 3).

3.2 Relative Safety and the Risk Premium

Flight to safety is a well documented phenomenon during periods of heightened risk, such as the Financial Crisis of 2007-2008 or the ongoing COVID-19 pandemic. Flight to safety seems intuitive when government bond is default free. But, how does a worsening of economic fundamentals of the safe asset affect its demand, and the demand for other assets?

To simplify the analysis, I assume $p = 1$ for this subsection. Suppose that investors can buy and sell shares in their projects to other investors at $t = 0$. Each share of any given project is a promise to deliver one unit of consumption in the no-default state and $\frac{a_L}{a_H - \delta}$ units of consumption if government defaults. Note that there are a total of $a_H - \delta$ of shares for any project. I assume that the investor who buys shares must pay the maintenance costs proportional to the amount of shares she holds in order to guarantee the promised payoffs. Since all projects are the same, investors can buy a share of the projects pool. Hence, for values of $\delta \le b_1 < \frac{\delta}{z}$, an investor solves:

$$\begin{cases} U = \max_{c_0,\theta_1,b_1} c_0 + \left\{(1-\pi_D)\ln(\theta_1 + b_1) + \pi_D \ln(\frac{a_L}{a_H-\delta}\theta_1 + zb_1)\right\}, \\ \\ s.t. \quad c_0 + q_0^a\theta_1 + q_0 b_1 = y_0 + q_0^a(a_H - \delta). \end{cases}$$

q_0^a is the price of a unit of share, and θ_1 is the number of shares purchased. Note that the equilibrium allocation is exactly the same as before, except that the project can be priced. Using the FOC with respect to θ_1, the market clearing condition $\theta_1 = a_H - \delta$, and that $\pi_D = F(b_1)$ in general equilibrium, I obtain:

$$\begin{cases} q_0 = \frac{1-F(b_1)}{a_H-\delta+b_1} + \frac{zF(b_1)}{a_L+zb_1}, \\ \\ q_0^a = \frac{1-F(b_1)}{a_H-\delta+b_1} + \frac{\frac{a_L}{a_H-\delta}F(b_1)}{a_L+zb_1}. \end{cases} \tag{5}$$

When $\frac{a_L}{z} < a_H - \delta$, it is clear from 5 that $q_0^a < q_0$. The reason is that project share is riskier than government bond. For any promised unit of consumption in the no-default state, a unit of bond and a share deliver z and $\frac{a_L}{a_H-\delta}$ in the default state, respectively. If $\frac{a_L}{a_H-\delta} < z$ (or equivalently $\frac{a_L}{z} < a_H - \delta$), shares pay less in the default state, are riskier and cheaper than bonds.

To study flight to safety, consider a negative shock to future tax revenues that changes the cumulative density to $\tilde{F}(b_1)$. Suppose that $\tilde{F}(b_1)$ first order stochastically dominates $F(b_1)$: $F(b_1) < \tilde{F}(b_1)$, for all b_1. If $\frac{a_L}{z} < a_H - \delta$, this negative shift *increases* the price of bond in 5, because $\frac{1}{a_H-\delta+b_1} < \frac{z}{a_L+zb_1}$. A negative shift in revenues also *decreases* the price of share given in 5, because $\frac{1}{a_H-\delta+b_1} > \frac{\frac{a_L}{a_H-\delta}}{a_L+zb_1}$. It follows that a negative shock to future tax revenues distribution increases the risk premium, $\frac{q_0}{q_0^a}$.

Although a worse prospect of future tax revenues distribution increases the default risk of the bond, such a negative shock increases both the bond price and the risk premium. The reason is that while bond becomes riskier as a result of a negative shock to future tax revenues, it remains the safer asset. The following lemma shows a similar result when the new income distribution is a mean-preserving spread of the old one:

Lemma 2. *Suppose that the future tax revenues distribution $f(b_1)$ changes to another symmetric triangular distribution $\tilde{f}(b_1)$. Assume that $\tilde{f}(b_1)$ has the same mean as $f(b_1)$, and the domain*

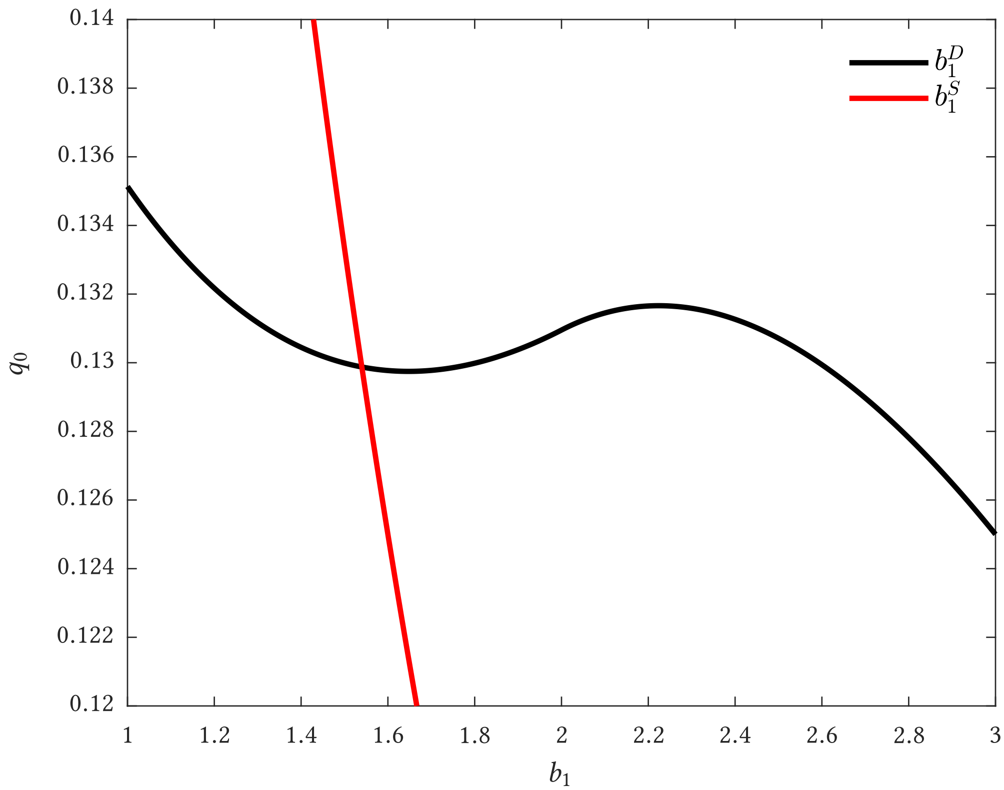

Figure 3: Supply and demand for government bonds. Demand curve is upward-sloping for the intermediate levels of government borrowing. Parameter values are $a_H = 7$, $a_L = 1$, $z = 0.2$, $\delta = 0.6$, $\tau \underline{y} = 1$, and $\tau \bar{y} = 3$.

$[\underline{b}", \overline{b}"]$, where $\underline{b}" = \underline{b} - \epsilon$ and $\overline{b}" = \overline{b} + \epsilon$. Then, the price of bond is higher and the price of share is lower under $\tilde{f}(b_1)$, for all $b_1 < \frac{\underline{b}+\overline{b}}{2}$.

Put simply, an increase in the risk without any change in the mean of the income distribution increases the bond price and decreases the price of share. As a result, an increase in risk also increases the risk premium.

The analysis so far assumed $\delta \leq b_1 < \frac{\delta}{z}$. When $b_1 \notin [\delta, \frac{\delta}{z})$, however, the above results do not hold. In particular, one can show that when b_1 is too high or too low, a first order stochastically dominant shift in $F(b_1)$ increases q_0^a and reduces q_0 and the risk premium, $\frac{q_0}{q_0^a}$. This implies that government bonds behave like a safe asset, when the relative supply of bonds to shares of the project in the aggregate, i.e., $\frac{b_1}{\theta_1}$, is within an intermediate range. Note that $\theta_1 = a_H - \delta$ in the benchmark model. Given that investors in this model can be thought of as banks or shadow banks, this also implies that government debt needs to be held in large enough quantities relative to the size of financial intermediaries' balance sheets in order to start behaving like a safe asset.

3.3 Upward-Sloping Demand Curve

The following conditions are necessary for an upward-sloping demand curve to exist:

Proposition 1. *Two necessary conditions for an upward-sloping demand curve are:*

$$\begin{cases} p \geq \frac{a_L(1-z)}{(a_H-a_L)z} \,, \\ \\ a_H - \delta > \frac{a_L}{z} \,. \end{cases}$$

Moreover, one has:

$$\frac{\partial q_0}{\partial b_1}|_{p=1} > 0 \Rightarrow \frac{\partial^2 q_0}{\partial p \partial b_1} > 0 \,.$$

The value of p is a measure of correlation between the payoffs of the two available assets, namely government bond and the project. When p is high, default by the government lowers the expected payoff to the project significantly, which increases the difference in expected marginal utilities of an extra unit of bond across default and no-default. This implies higher aggregate risk and increases the demand for insurance by investors and the price of bond.

Given z, the second necessary condition in Proposition 1 states that a_L must be low enough relative to a_H or, in other words, government default should inflict a sizable damage to the rest of the economy. As mentioned before, the second necessary condition also implies that z can not be too low. Note that default in this model can capture both a fiscal default and inflation.

The meaning of the second result in Proposition 1 is as follows. When demand for bond is upward sloping at $p = 1$, lowering p decreases the slope of the demand curve: it is less likely to have an upward-sloping demand curve for lower values of p.

It is important to note that whenever the demand curve for b_1 is upward sloping, an increase in government expenditures g_0 increases the equilibrium price of government bonds. A positive relationship between b_1 and bond price q_0 implies a positive relationship between $g_0 = q_0 b1$ and bond price: the demand curve is upward sloping no matter how one measures government debt. I state this simple but important observation in the following lemma.

Lemma 3. *If* $\frac{\partial q_0}{\partial b_1} \geq 0$ *in equilibrium, then* $\frac{\partial q_0}{\partial g_0} \geq 0$.

To construct an example of an upward-sloping demand curve for government bonds, I assume that $f(b_1)$ is a triangular distribution (Figure 2). Later, I discuss what can be shown for somewhat more general class of distributions. Suppose that $f(b_1)$ is a symmetric triangular distribution over

$[\underline{b}, \overline{b}]$, which is defined as follows:

$$f(b_1) \equiv \begin{cases} \frac{4(b_1 - \underline{b})}{(\overline{b} - \underline{b})^2} & b_1 \in [\underline{b}, \frac{\underline{b}+\overline{b}}{2}] \,, \\ \\ \frac{4(\overline{b} - b_1)}{(\overline{b} - \underline{b})^2} & b_1 \in [\frac{\underline{b}+\overline{b}}{2}, \overline{b}] \,. \end{cases} \tag{6}$$

where $\underline{b} \equiv \tau \underline{y}$ and $\overline{b} \equiv \tau \overline{y}$ are the minimum and maximum values of tax collected by the government at $t = 1$. I also make the following assumptions to ensure that tax receipts are always enough to pay the default value to the investors, and that $\delta \leq b_1 \leq \frac{\delta}{z}$:

Assumption 2. *$z\overline{y} < \underline{y}$, $\delta < \underline{b}$, and $\overline{b} < \frac{\delta}{z}$.*

From now on, I also set $p = 1$ to further simplify the analysis in this and subsequent sections. The cumulative density function for $b_1 \in [\underline{b}, \frac{\underline{b}+\overline{b}}{2}]$ is:

$$F(b_1) = 2\left(\frac{b_1 - \underline{b}}{\overline{b} - \underline{b}}\right)^2 . \tag{7}$$

For all $b_1 \in [\underline{b}, \frac{\underline{b}+\overline{b}}{2}]$, 1 implies:

$$\frac{\partial q_0}{\partial b_1} \propto \frac{a_H - \frac{a_L}{z} - \delta}{\frac{a_L}{z} + b_1} \left(\frac{b_1 - \underline{b}}{\overline{b} - \underline{b}}\right)^2 \left\{ \frac{2}{b_1 - \underline{b}} - \left[\frac{1}{\frac{a_L}{z} + b_1} + \frac{1}{a_H - \delta + b_1} \right] \right\} - \frac{1}{a_H - \delta + b_1} . \tag{8}$$

The term in the braces in 8 is always positive. Moreover, for values of b_1 close to $\underline{b}$, the first term in the expression on the RHS is close to zero, which implies that the slope of the demand curve is unambiguously negative. But, for a given a value of $b_1 \in [\underline{b}, \frac{\underline{b}+\overline{b}}{2}]$, a large enough value of a_H implies a positive slope of demand curve at b_1:

Proposition 2. *For any $\tilde{b}_1 \in (\underline{b}, \frac{\underline{b}+\overline{b}}{2}]$, if $a_H - \delta - \frac{a_L}{z}$ is large enough, one has $\frac{\partial q_0}{\partial b_1} > 0$ for all $b_1 \in [\tilde{b}_1, \frac{\underline{b}+\overline{b}}{2}]$. Moreover, for given values of a_H and a_L, there exists $\hat{b}_1 > \underline{b}$ such that $\frac{\partial q_0}{\partial b_1} < 0$, for all $b_1 \in [\underline{b}, \tilde{b}_1]$.*

For low values of borrowing, the demand curve is downward sloping. But, there is a point at which the slope of the demand curve turns positive, and stays positive at least until the mean of the distribution, $\frac{\underline{b}+\overline{b}}{2}$. The reason behind an upward-sloping demand curve is that what matters is not the absolute safety of the bond but its riskiness *relative* to other investments. In other words, government bond is a safe asset because it remains safer than the share even when default risk goes up.

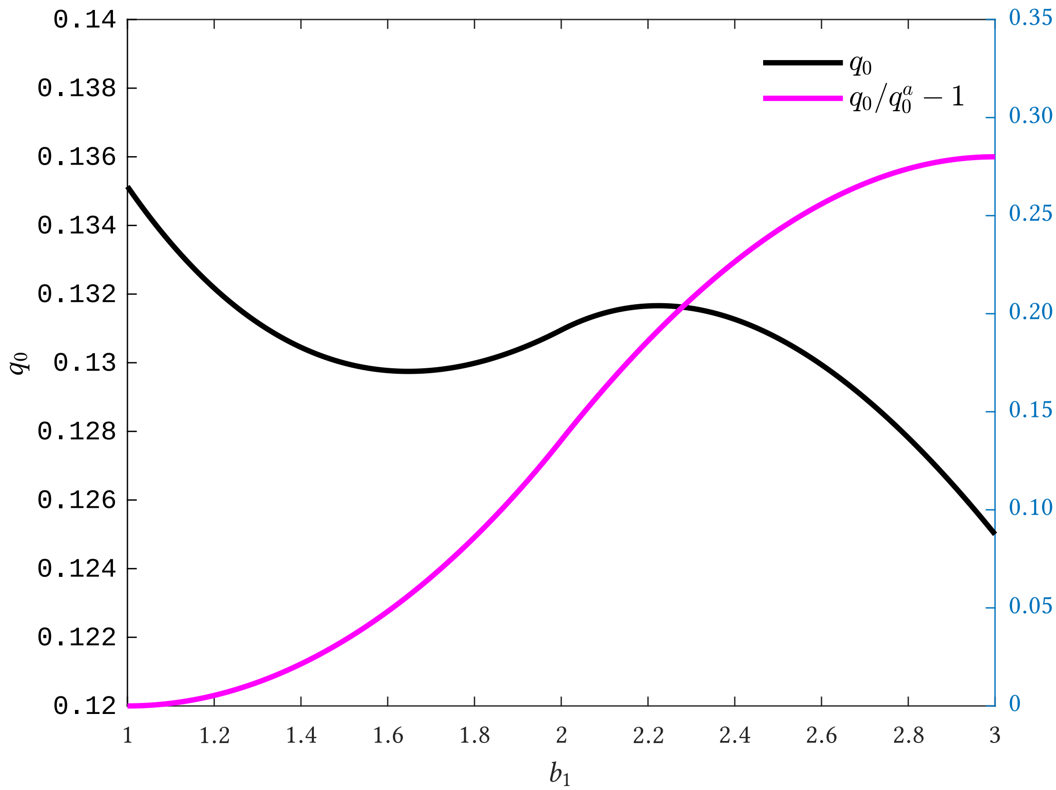

Figure 4: Demand for government bonds and the risk premium. Demand curve is upward-sloping for the intermediate levels of government borrowing. Parameter values are $a_H = 7$, $a_L = 1$, $z = 0.2$, $\delta = 0.6$, $\tau \underline{y} = 1$, and $\tau \bar{y} = 3$.

A higher value of $a_H - \delta - \frac{a_L}{z}$ can reflect more riskiness of the project's payoff. Higher riskiness can be due to a higher a_H, a lower a_L, or a higher p. As discussed before, examples of project in this model include investments by banks and shadow banks. One factor that can increase the riskiness of these investments in the real world is high leverage of both borrowers and lenders. More than a decade after the Financial Crisis of 2008, banks and many shadow banks remain highly leveraged entities. The non-financial corporate debt as a fraction of GDP has been on the rise in Euro area, the U.S., and China.[4] Although borrowers and lenders are not explicitly modeled, one can think of a higher leverage of banks and shadow banks, i.e., investors, to be reflected in a higher upside a_H. A higher leverage in the household or the corporate sector, on the other hand, can be captured by a higher p: liquidity problems are more likely to become solvency problems when firms have high leverage. I study the effect of investors' indebtedness more explicitly in Section 4.

Figure 3 is an illustration of an partially upward-sloping demand curve for government bonds. In Lemma 1, I have already shown that demand for government debt is downward sloping when

[4]See April 2019 issue of the Global Financial Stability Report, published by IMF.

$b_1 \notin [\delta, \frac{\delta}{z})$. But, Figure 3 shows that demand curve becomes downward sloping in the left and right tails of the domain of b_1, even when $b_1 \in [\delta, \frac{\delta}{z})$. I show in the next subsection that this property is rather general and does not depend on the particular distribution used in this subsection.

Liu (2019) finds that an increase in U.S. government debt is associated with lower short term rate. Cochrane (2020) also finds that large amounts of debt correspond to low subsequent returns. These findings are consistent with an upward-sloping demand curve for U.S. debt.

How does an increase in borrowing affect the risk premium, i.e., price of bond relative to share? In general, the risk premium, or $\frac{q_0}{q_0^a}$, can increase or decrease with respect to an increase in b_1. But, one can be more specific under upward-sloping demand curve:

Lemma 4. *$\frac{q_0}{q_0^a}$ is increasing in b_1, if $\frac{\partial q_0}{\partial b_1} \geq 0$.*

When the demand for bond is upward sloping, an increase in default risk increases the risk premium (Figure 4). This is intuitive because, facing with a higher level of default risk, investors value safety even more. This causes a shift in their demand for bonds relative to shares, which increases their relative price, $\frac{q_0}{q_0^a}$, in equilibrium.

Krishnamurthy and Vissing-Jorgensen (2012) find that the spread between AAA-rated corporate bonds and U.S. treasury is negatively correlated with the debt to GDP ratio. They show that both the safety and liquidly premium of U.S. Treasuries contribute to the movements in this spread. However, Liu (2019) finds that an increase in U.S. government debt predicts higher risk premia in equity and bond markets. The behavior of risk premium in this model is consistent with Liu (2019) under an upward-sloping demand curve.

3.4 General Income Distributions

Using 1 at $p = 1$, I obtain:

$$\frac{\partial q_0}{\partial b_1} \propto \frac{(a_H - \delta - \frac{a_L}{z})F(b_1)}{\frac{a_L}{z} + b_1}\left\{\frac{f(b_1)}{F(b_1)} - \left[\frac{1}{\frac{a_L}{z} + b_1} + \frac{1}{a_H - \delta + b_1}\right]\right\} - \frac{1}{a_H - \delta + b_1}$$

Since $\frac{a_L}{z} + b_1 < a_H - \delta + b_1$, a sufficient condition for $\frac{\partial q_0}{\partial b_1} > 0$ is $\zeta(b_1) > 1$, where:

$$\zeta(b_1) = (a_H - \delta - \frac{a_L}{z})F(b_1)\left\{\frac{f(b_1)}{F(b_1)} - \left[\frac{1}{\frac{a_L}{z} + b_1} + \frac{1}{a_H - \delta + b_1}\right]\right\}$$

Consider a continuous distribution $f(b_1)$, which has a bounded support $[\underline{b}, \overline{b}]$. If $f(\underline{b}) > 0$, it is clear that $\zeta(b_1) > 0$ for b_1 close to $\underline{b}$, as $F(b_1)$ is arbitrarily close to zero. In such a case, a high enough $a_H - \delta - \frac{a_L}{z}$ implies that $\zeta(b_1) > 1$ and consequently $\frac{\partial q_0}{\partial b_1} > 0$.

Now consider the case $f(\underline{b}) = f(\overline{b}) = 0$. A necessary condition for $\frac{\partial q_0}{\partial b_1} > 0$ is that the expression in braces in the definition of $\zeta(b_1)$, is positive. The first term in the braces, which is the reversed hazard function $\frac{f(b_1)}{F(b_1)}$, is the derivative of $\ln(F(b_1))$. Hence, the value of $\frac{f(b_1)}{F(b_1)}$ depends on the convexity of $f(b_1)$. Therefore, I can state the following result:

LEMMA 5. *Suppose that $f(b_1)$ is a continuous density function over $[\underline{b}, \overline{b}]$ that satisfies $f(\underline{b}) = f(\overline{b}) = 0$, and $f(b_1) > 0$ for $(\underline{b}, \overline{b})$. Assume that $f(b_1)$ is convex for $b_1 \in [\underline{b}, \hat{b}]$, for some $\underline{b} < \hat{b} \leq \overline{b}$. Then, for a large enough $a_H - \delta - \frac{a_L}{z}$, one has $\frac{\partial q_0}{\partial b_1}|_{\hat{b}} > 0$.*

Note that $\frac{\partial q_0}{\partial b_1} < 0$ for values of b_1 close to either $\underline{b}$ or $\overline{b}$, since $f(\underline{b}) = f(\overline{b}) = 0$. Therefore, b_1 needs to be within an intermediate range, i.e., neither too low nor too high, for the demand curve to be upward-sloping. This also implies that an upward-sloping segment for intermediate values of b_1 always comes with (at least) two inflection points. The lower inflection point is the value of b_1, where the price of government debt starts to *increase* with more supply. And the second inflection point is the value of b_1 at which, the price of government debt starts to *decrease* with more supply again.

Lemma 5 implies that as $a_H - \delta - \frac{a_L}{z}$ becomes larger, a larger neighborhood of $\hat{b}$ and, consequently, a larger fraction of the interval $b_1 \in [\underline{b}, \hat{b}]$ exhibits upward-sloping demand curve.

When $\frac{a_L}{z} < a_H - \delta$, a marginal increase in borrowing by the government increases the probability of default by $f(b_1)$, which increases the demand for insurance and the risk premium on government bonds. This is the *change in aggregate risk* effect. At the same time, such an increase in borrowing reduces the demand for government bond due to the *diminishing expected marginal utility* effect. $\zeta(b_1) > 0$ is a necessary condition for the net of these two effects to be positive.

As can be seen in the expression for $\zeta(b_1)$, one determinant of the net impact of these two effects is the reversed hazard function $\frac{f(b_1)}{F(b_1)}$. When $f(b_1)$ is convex at its left tale, an increase in b_1 increases $\frac{f(b_1)}{F(b_1)}$. This makes it more likely to have an upward-sloping demand curve, for high enough b_1. Many unimodal distributions, such as normal, log-normal and Weibull distributions, are convex on their left tale.[5] But, note that convexity is a sufficient and not a necessary condition for an upward-sloping demand curve.

[5]The left-tale of Weibull density function is convex for $k > 2$, where k is the shape parameter.

3.5 Access to Other Safe Assets

In my benchmark model, I have assumed that investors do not have access to any other private or public safe assets. But, in the real world, investors may have access to more than one type of safe asset. Alternatively, and without explicitly modeling a second government, suppose that a second government bond is available to investors (let's call the two assets first and second government bonds). If default risk of the first and second second government bond are significantly correlated, my main results hold. The reason is that investing in the second bond cannot provide a better hedge against aggregate risk. This might be the case, for instance, when the two issuing governments have significant bilateral trade and financial flows and cross holdings of each other's assets including their government bonds. In such a case, default risk can become even more correlated since large increases in government debts might become synchronized.[6]

Alternatively, assume that this second bond is risk free and that its fixed supply is equal to b^{rf*}. In this case, the demand curve for b_1 is given by Lemma 1, and its slope is given by 2, given $a_L^* = a_L + b^{rf*}$ and $a_H^* = a_H + b^{rf*}$. In order for the first government bond to remain safer than the project, one needs to have:

$$a_H^* - \delta > \frac{a_L^*}{z} \iff b^{rf*} < \frac{(a_H - \delta - \frac{a_L}{z})z}{1-z}.$$

Therefore, as long as b^{rf*} is not too large, the results in the previous subsections hold. This condition is based on the assumption of fixed supply, but it can also be a useful approximation when the supply of the second government bond is inelastic enough.

But, why should access to other government bonds be limited? In the case of banks, there are various reasons why they might prefer or even be encouraged to hold own government debt. These include especial treatment of own government bonds by capital and liquidity regulations, central banks' liquidity operations, risk taking, and financial repression (Dell'Ariccia et al. (2018)).

4 Balance-Sheet Effects

Another potentially important mechanism that can produce an upward-sloping demand curve and flight to safety for a risky government bond is when investors hold a significant amount of *old* government debt, that is partially financed by borrowing. Consider the model from the previous sections with the same agents. The investors come into period $t = 0$ with an outstanding amount of government bonds, $\tilde{b}_1$, and an outstanding level of liabilities to the household, d_1, both maturing at $t = 1$. One can think of $\tilde{b}_1$ as long-term bonds purchased in a previous period, which has been

[6]This has been the case, for instance, during the Financial Crisis of 2007-2008 and the ongoing COVID-19 recession.

partially financed by d_1. Similar to the benchmark model, one can think of investors as banks or shadow banks. d_1 can be thought as a proxy for leverage and captures different types of funding used by banks and shadow banks, including deposits, repo, and commercial papers. I assume that a government default affects the payments of both the old and new bonds alike. Suppose that the default value of the outstanding bonds is enough to pay the outstanding liabilities, or $d_1 < z\tilde{b}_1$. As before, the future tax receipts are distributed according to a cumulative density F, which is symmetric and triangular with the domain $[\underline{b}, \overline{b}]$.

As before, the FOC and the probability of default as a function of $b_1 + \tilde{b}_1$ yield the bond price in equilibrium:

$$q_0(\tilde{b}_1, d_1; b_1) = \frac{1 - F(\tilde{b}_1 + b_1)}{(\tilde{b}_1 - d_1) + b_1} + \frac{zF(\tilde{b}_1 + b_1)}{(z\tilde{b}_1 - d_1) + zb_1} \,. \tag{9}$$

Note that if $d_1 = 0$, the price of bond in 9 is equal to $\frac{1}{\tilde{b}_1 + b_1}$, which is strictly decreasing in $\tilde{b}_1$ and b_1. Therefore, in order to have an upward-sloping demand curve, one must have $d_1 > 0$. $\tilde{b}_1$ and b_1, enter the price equation as a sum, which facilitates the following comparative statics:

Proposition 3. *Suppose $d_1 \leq z\underline{b}$, and that:*

$$\frac{1-z}{z} d_1 \left\{ \frac{1}{\overline{b} - \underline{b}} - \frac{1}{\overline{b} + (1-2z)\underline{b}} \right\} \geq 1 \,.$$

Then, for large enough value of $\tilde{b}_1 + b_1 \in [\underline{b}, \frac{\underline{b}+\overline{b}}{2}]$:

$$\begin{cases} \dfrac{\partial q_0}{\partial \tilde{b}_1} = \dfrac{\partial q_0}{\partial b_1} > 0 \,, \\[4em] \dfrac{\partial^2 q_0}{\partial d_1 \partial \tilde{b}_1} > 0 \,. \end{cases}$$

One case, in which the condition in Proposition 3 is satisfied, is when $\underline{b}$ is large enough. The intuition for $\frac{\partial q_0}{\partial \tilde{b}_1} = \frac{\partial q_0}{\partial b_1} > 0$ is similar to the previous sections. The effects of an increase in $b_1 + \tilde{b}_1$ are twofold. On the one hand, such an increase yields more consumption in both future states. This effect depresses the price because of a lower expected marginal utility of an extra unit of bond. On the other hand, an increase in $b_1\tilde{b}_1$ creates more risk of default, and aggregate risk. This latter effect, similar to the previous case, creates more demand for insurance. Whenever the second effect is dominant, an increase in initial bond holding increases the price of new bonds.

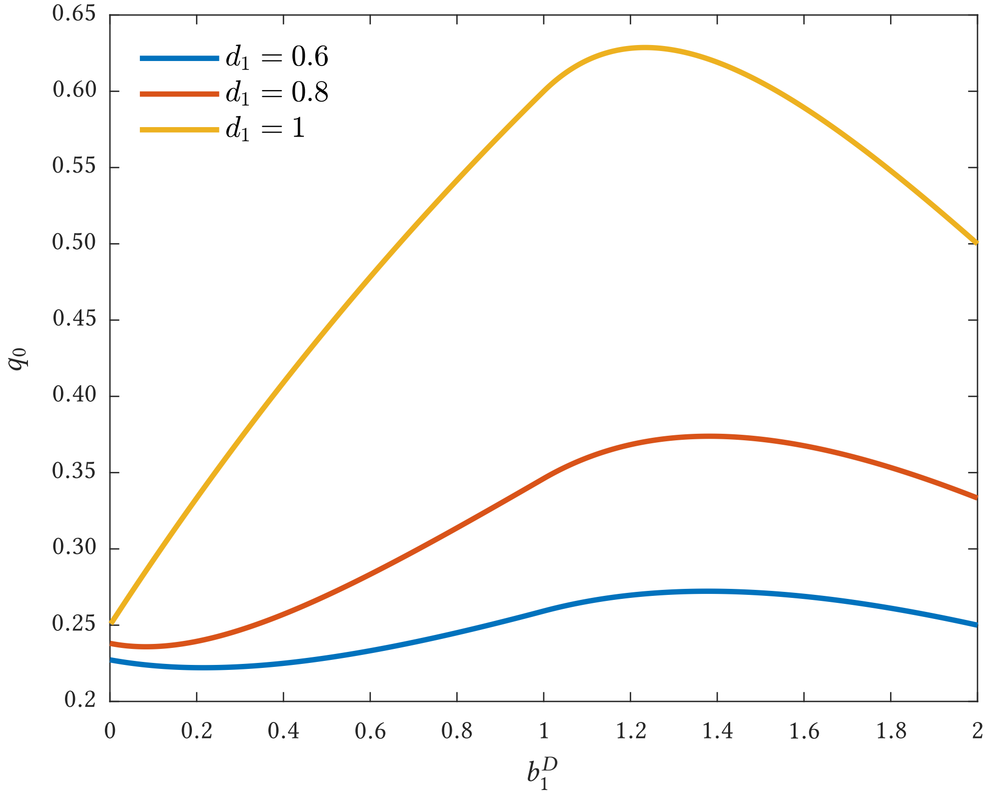

Figure 5: Demand for government bonds for different values of d_1. Other parameter values are $\tilde{b}_1 = 5$, $z = 0.2$, $\delta = 1$, $\tau \underline{y} = 5$, and $\tau \bar{y} = 7$.

The fact that $b_1 + \tilde{b}_1$ should be high enough in Proposition 3, implies that a high enough level of old government debt can make an upward-sloping demand curve more likely. In such a case, and as before, a higher level of b_1 leads to lower real interest rate (higher q_0) and higher aggregate default risk.

The last result in Proposition 3 states that, for high enough value of debt, larger investors' liabilities make an upward-sloping demand curve more likely (Figure 4). The intuition of this result is as follows. The default value of the old investments, including assets $\tilde{b}_1$ and liabilities d_1, per unit of return in the normal future state is $\frac{z\tilde{b}_1 - d_1}{\tilde{b}_1 - d_1}$. This ratio is lower than z, and is decreasing in d_1. This implies that when d_1 is large, the differential riskiness between investment in new bonds and the old bonds is larger. Therefore, with a higher d_1, an increase in b_1 increases risk and the demand for insurance even more.

5 Conclusion

In this paper, I study the demand for risky government bonds as safe assets. I show that when government default significantly affects the return to other available assets, government bonds become the safe asset. An exogenous deterioration in the future prospects, in the form of a negative shift or a mean preserving spread of the future tax revenues distribution, increases the price of government bonds and decreases the price of other assets, e.g., shares in the benchmark model. This implies that such an exogenous increase in risks also increases the risk premium.

I also show that government bonds can face an upward-sloping demand curve, when they are safe assets. This happens when the increase in default risks due to higher levels of issuance, significantly increases the demand for insurance. I show that, whether this is the case, depends on the return portfolio of the other assets and the reversed hazard rate of the future tax revenues distribution.

While the model is stylized, one can extend it to a fully dynamic infinite horizon setup. Such extension can be useful in estimating the contribution of the mechanisms discussed in this paper to the evolution of the real rates and risk premia. The model can have welfare implications for the conduct of fiscal policy, when the economy is at the zero lower bound. If nominal frictions make the real rate sticky, a fiscal expansion that lowers the equilibrium real interest rate, can depress the aggregate demand and welfare. These and other potentially interesting extensions of the model can be subjects for future research.

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A Appendix: Proofs

Proof of Lemma 1. I derived the demand curve for the case $\delta \leq b_1 < \frac{\delta}{z}$. I assume for the moment that investors pay the maintenance costs if they can. I derive the demand and show it is downward sloping only for the case $b_1 \geq \frac{\delta}{z}$, as the third case $b_1 < \delta$ is very similar. When $b_1 \geq \frac{\delta}{z}$, investors pay the maintenance costs in all future states. Therefore, the payoffs in the default and no-default states are $a_H - \delta + zb_1$ and $a_H - \delta + b_1$, respectively. Thus, the demand function immediately follows from the FOCs of the investors. For $b_1 \geq \frac{\delta}{z}$, taking the derivative with respect to b_1, I obtain:

$$\frac{\partial q_0}{\partial b_1} = -\Big\{\Big[\frac{1-F(b_1)}{(a_H-\delta+b_1)^2} + \frac{zF(b_1)}{(a_H-\delta+zb_1)^2}\Big] + \Big[\frac{1}{a_H-\delta+b_1} - \frac{z}{a_H-\delta+zb_1}\Big]f(b_1)\Big\} < 0\,.$$

Note that both terms inside the braces are strictly positive. Next, I need to show that given Assumption 1, investors choose to pay the maintenance costs if they can. Investors pay the maintenance costs, under no default, if:

$$\ln(a_H-\delta+b_1) \geq p\ln(a_L+b_1) + (1-p)\ln(a_H+b_1) \iff \frac{a_H-\delta+b_1}{a_H+b_1} \geq \Big(\frac{a_L+b_1}{a_H+b_1}\Big)^p\,.$$

Define the function $\gamma(b_1)$ as the difference between the LHS and the RHS of the first inequality above:

$$\gamma(b_1) \equiv \ln(a_H-\delta+b_1) - (p\ln(a_L+b_1) + (1-p)\ln(a_H+b_1)) \Rightarrow$$

$$\Rightarrow \frac{\partial\gamma}{\partial b_1} = \frac{1}{a_H+b_1}\Big(\frac{\delta}{a_H-\delta+b_1} - \frac{p(a_H-a_L)}{a_L+b_1}\Big)\,.$$

Under Assumption 1, the term inside the parentheses is strictly negative, hence, $\frac{\partial\gamma}{\partial b_1} < 0$. This implies that if $\gamma(b_1) > 0$, then $\gamma(\tilde{b}_1) > 0$ for all $\tilde{b}_1 < b_1$, and specifically $\gamma(zb_1) > 0$. Therefore, it suffices to find a lower bound for p that works under the no default state. Using the second inequality above, I obtain:

$$\frac{a_H-\delta+b_1}{a_H+b_1} \geq \Big(\frac{a_L+b_1}{a_H+b_1}\Big)^p \iff p \geq \frac{\ln(a_H+b_1)-\ln(a_H-\delta+b_1)}{\ln(a_H+b_1)-\ln(a_L+b_1)} \equiv h(b_1)\,.$$

Taking the derivative, one has:

$$\frac{\partial h}{\partial b_1} = \frac{\Big(\frac{1}{a_H+b_1} - \frac{1}{a_H-\delta+b_1}\Big)(\ln(a_H+b_1)-\ln(a_L+b_1)) - \Big(\frac{1}{a_H+b_1} - \frac{1}{a_L+b_1}\Big)(\ln(a_H+b_1)-\ln(a_H-\delta+b_1))}{(\ln(a_H+b_1)-\ln(a_L+b_1))^2} > 0\,.$$

Since $h(b_1)$ is an increasing function, if $h(b_1)$ has a limit as $b_1 \to \infty$, that limit can provide a sufficient condition on p. Both the numerator and the denominator of $h(b_1)$ go to zero as $b_1 \to \infty$. The limit, using the L'Hopital's rule is given by:

$$\lim_{b_1\to\infty} h(b_1) = \lim_{b_1\to\infty} \frac{\frac{1}{a_H+b_1} - \frac{1}{a_H-\delta+b_1}}{\frac{1}{a_H+b_1} - \frac{1}{a_L+b_1}} = \frac{\delta}{a_H - a_L} \lim_{b_1\to\infty} \frac{a_L + b_1}{a_H - \delta + b_1} = \frac{\delta}{a_H - a_L}\,.$$

Therefore, if $p \geq \frac{\delta}{a_H - a_L}$, investors pay for the maintenance costs if they can. ■

Proof of Lemma 2. Note that:

$$\begin{cases} \frac{1}{a_H-\delta+b_1} < \frac{1}{\frac{a_L}{z}+b_1}\,, \\ \\ \frac{a_H-\delta}{a_H-\delta+b_1} > \frac{\frac{a_L}{z}}{\frac{a_L}{z}+b_1}\,. \end{cases}$$

Therefore, it suffices to show that $\tilde{F}(b_1) > F(b_1)$ for all $b_1 \in [\underline{b}, \frac{\underline{b}+\overline{b}}{2})$. To see this, note that:

$$\tilde{F}(b_1) > F(b_1) \iff \frac{b_1 - \underline{b} + \epsilon}{\overline{b} - \underline{b} + 2\epsilon} > \frac{b_1 - \underline{b}}{\overline{b} - \underline{b}} \iff \overline{b} - \underline{b} > 2(b_1 - \underline{b}) \iff b_1 < \frac{\underline{b} + \overline{b}}{2}\,.$$

■

Proof of Proposition 1. I can rewrite the expression in 2 as follows:

$$\frac{\partial q_0}{\partial b_1} = -\frac{1 - F(b_1)}{(a_H - \delta + b_1)^2} - \Big[\frac{(1-p)z^2}{(a_H + zb_1)^2} + \frac{pz^2}{(a_L + zb_1)^2}\Big]F(b_1) +$$

$$+ \Big[\frac{(1-p)z}{a_H + zb_1} + \frac{pz}{a_L + zb_1} - \frac{1}{a_H - \delta + b_1}\Big]f(b_1)\,.$$

It is clear from the above that the first two terms in the first line above are negative. If $a_H - \delta \leq \frac{a_L}{z}$, the third term in the second line would be non-positive regardless of the value of p, implying that $\frac{\partial q_0}{\partial b_1} < 0$. Therefore $a_H - \delta > \frac{a_L}{z}$ is a necessary condition for an upward-sloping demand curve. To have an upward-sloping demand curve, the last term in the expression for $\frac{\partial q_0}{\partial b_1}$ above needs to be strictly positive. This is the case iff:

$$\frac{(1-p)z}{a_H+zb_1}+\frac{pz}{a_L+zb_1}-\frac{1}{a_H-\delta+b_1}>0 \iff \Big(\frac{1}{\frac{a_L}{z}+b_1}-\frac{1}{a_H-\delta+b_1}\Big)-\Big(\frac{1}{\frac{a_L}{z}+b_1}-\frac{1}{\frac{a_H}{z}+b_1}\Big)(1-p)>0\,,$$

$$\iff 1-p<\frac{z(a_H-\delta-\frac{a_L}{z})}{a_H-a_L}\frac{\frac{a_H}{z}+b_1}{a_H-\delta+b_1}\,.$$

But, $\frac{\frac{a_H}{z}+b_1}{a_H-\delta+b_1}<\frac{\frac{a_H}{z}}{a_H-\delta}$, as the numerator is larger than the denominator. Therefore, one has:

$$\frac{\partial q_0}{\partial b_1}>0 \Rightarrow 1-p<\frac{z(a_H-\delta-\frac{a_L}{z})}{a_H-a_L}\frac{\frac{a_H}{z}}{a_H-\delta}=\frac{a_H}{a_H-a_L}\frac{a_H-\delta-\frac{a_L}{z}}{a_H-\delta}<\frac{a_H-\frac{a_L}{z}}{a_H-a_L}\Rightarrow,$$

$$\Rightarrow p>\frac{a_L(1-z)}{(a_H-a_L)z}\,.$$

To see the last result, note that we can collect all terms multiplying $1-p$, and rewrite the expression for $\frac{\partial q_0}{\partial b_1}$ as follows:

$$\frac{\partial q_0}{\partial b_1}=-\Big(\frac{1-F(b_1)}{(a_H-\delta+b_1)^2}+\frac{f(b_1)}{a_H-\delta+b_1}\Big)+\frac{f(b_1)}{\frac{a_L}{z}+b_1}-\frac{F(b_1)}{(\frac{a_L}{z}+b_1)^2}+$$

$$+\Big\{\Big[\frac{1}{(\frac{a_L}{z}+b_1)^2}-\frac{1}{(\frac{a_H}{z}+b_1)^2}\Big]F(b_1)-\Big[\frac{1}{\frac{a_L}{z}+b_1}-\frac{1}{\frac{a_H}{z}+b_1}\Big]f(b_1)\Big\}(1-p)\,.$$

But, one has:

$$\frac{\partial q_0}{\partial b_1}|_{p=1}=-\frac{1}{(a_H-\delta+b_1)^2}+\Big[f(b_1)-\Big(\frac{1}{\frac{a_L}{z}+b_1}+\frac{1}{a_H+b_1}\Big)F(b_1)\Big]\Big(\frac{1}{\frac{a_L}{z}+b_1}-\frac{1}{a_H+b_1}\Big)$$

If $\frac{\partial q_0}{\partial b_1}|_{p=1}>0$, it must be that $f(b_1)-\Big(\frac{1}{\frac{a_L}{z}+b_1}+\frac{1}{a_H+b_1}\Big)F(b_1)>0$. This implies that the term that is multiplied by $1-p$ in the expression for $\frac{\partial q_0}{\partial b_1}$ must be strictly negative as $\frac{1}{\frac{a_L}{z}+b_1}+\frac{1}{\frac{a_H}{z}+b_1}<\frac{1}{\frac{a_L}{z}+b_1}+\frac{1}{a_H+b_1}$. This implies:

$$\frac{\partial q_0}{\partial b_1}|_{p=1}>0\Rightarrow\frac{\partial^2 q_0}{\partial p\partial b_1}>0\,.$$

■

Proof of Lemma 3. One has:

$$\frac{\partial q_0}{\partial g_0} = \frac{\frac{\partial q_0}{\partial b_1}}{\frac{\partial g_0}{\partial b_1}} = \frac{\frac{\partial q_0}{\partial b_1}}{q_0 + \frac{\partial q_0}{\partial b_1} b_1} .$$

It is clear from the above that if $\frac{\partial q_0}{\partial b_1} \geq 0$, one has $\frac{\partial q_0}{\partial g_0} \geq 0$. ∎

Proof of Proposition 2. For the triangular distribution, the expression for $\frac{\partial q_0}{\partial b_1}$ in 2, for $p = 1$ and $b_1 \in [\underline{b}, \frac{\underline{b}+\overline{b}}{2}]$ becomes:

$$\frac{\partial q_0}{\partial b_1} = \frac{1}{a_H - \delta + b_1} \left\{ \frac{a_H - \frac{a_L}{z} - \delta}{\frac{a_L}{z} + b_1} \left(\frac{b_1 - \underline{b}}{\overline{b} - \underline{b}} \right)^2 \left\{ \frac{2}{b_1 - \underline{b}} - \left[\frac{1}{\frac{a_L}{z} + b_1} + \frac{1}{a_H - \delta + b_1} \right] \right\} - \frac{1}{a_H - \delta + b_1} \right\} .$$

Note that the term in the first pair of inner braces is strictly positive. Suppose that $a_H - \frac{a_L}{z} - \delta$ is large enough, such that:

$$\lambda(\tilde{b}_1) \equiv (a_H - \frac{a_L}{z} - \delta) \left(\frac{\tilde{b}_1 - \underline{b}}{\overline{b} - \underline{b}} \right)^2 \left\{ \frac{2}{\tilde{b}_1 - \underline{b}} - \left[\frac{1}{\frac{a_L}{z} + \tilde{b}_1} + \frac{1}{a_H - \delta + \tilde{b}_1} \right] \right\} \geq 1 .$$

The derivative of $\lambda(\tilde{b}_1)$ is:

$$\frac{\partial \lambda}{\partial \tilde{b}_1} \propto \left(1 - \frac{\tilde{b}_1 - \underline{b}}{\frac{a_L}{z} + \tilde{b}_1} \right)^2 + \left(1 - \frac{\tilde{b}_1 - \underline{b}}{a_H - \delta + \tilde{b}_1} \right)^2 > 0 .$$

Since $\frac{1}{\frac{a_L}{z} + b_1} > \frac{1}{a_H - \delta + b_1}$ and that $\lambda(b_1) \geq 1$ for all $b_1 \in [\tilde{b}_1, \frac{\underline{b}+\overline{b}}{2}]$, one has $\frac{\partial q_0}{\partial b_1} > 0$ for all $b_1 \in [\tilde{b}_1, \frac{\underline{b}+\overline{b}}{2}]$. For the last part of the proof, it suffices to note that $\lambda(b_1) \to 0$, as $b_1 \to \underline{b}$. ∎

Proof of Lemma 4. Using 5, I obtain:

$$\frac{q_0^a}{q_0} = a_H - \delta - (a_H - \frac{a_L}{z} - \delta) \frac{\frac{zF(b_1)}{a_L + zb_1}}{\frac{zF(b_1)}{a_L + zb_1} + \frac{1 - F(b_1)}{a_H - \delta + b_1}} .$$

Define:

$$\zeta_1(b_1) \equiv \frac{\frac{zF(b_1)}{a_L + zb_1}}{\frac{zF(b_1)}{a_L + zb_1} + \frac{1 - F(b_1)}{a_H - \delta + b_1}} .$$

Then, one has:

$$\frac{1}{\zeta_1(b_1)} = 1 + \frac{(1 - F(b_1))(\frac{a_L}{z} + b_1)}{F(b_1)(a_H - \delta + b_1)} .$$

If I define $\zeta_2(b_1) \equiv \frac{(1-F(b_1))(\frac{a_L}{z}+b_1)}{F(b_1)(a_H-\delta+b_1)}$, then:

$$\frac{\partial \zeta_2}{\partial b_1} \propto (a_H - \frac{a_L}{z} - \delta)(1-F(b_1))F(b_1) - (a_H - \delta + b_1)(\frac{a_L}{z} + b_1)f(b_1) =$$

$$-(a_H - \delta + b_1)(\frac{a_L}{z} + b_1)\Big\{f(b_1) - \Big[\frac{1}{\frac{a_L}{z} + b_1} - \frac{1}{a_H - \delta + b_1}\Big](1-F(b_1))F(b_1)\Big\}.$$

From the proof of Proposition 1, one knows that if $\frac{\partial q_0}{\partial b_1}|_{p=1} > 0$, it must be that $f(b_1) - \Big(\frac{1}{\frac{a_L}{z}+b_1} + \frac{1}{a_H+b_1}\Big)F(b_1) > 0$. This immediately implies that, if $\frac{\partial q_0}{\partial b_1}|_{p=1} > 0$ then $\frac{\partial \zeta_2}{\partial b_1} < 0$. But, if $\zeta_2(b_1)$ is decreasing in b_1, $\zeta_1(b_1)$ must be increasing, and $\frac{q_0^a}{q_0}$ ($\frac{q_0}{q_0^a}$) must be decreasing (increasing) in b_1. ■

Proof of Lemma 5. Since $f(b_1)$ is convex for $b_1 \in (\underline{b}, \hat{b})$, one has:

$$f(b_1) \leq \lambda f(\underline{b}) + (1-\lambda)f(\hat{b}).$$

Where, $b_1 = \lambda\underline{b} + (1-\lambda)\hat{b}$. This implies that the area under $f(b_1)$ between $\underline{b}$ and $\hat{b}$, which is equal to $F(\hat{b})$, is not larger than the area of the triangle with the base $\hat{b} - \underline{b}$ and height of $f(\hat{b})$. Therefore, one has:

$$\frac{f(\hat{b})}{F(\hat{b})} \geq \frac{f(\hat{b})}{\frac{1}{2}(\hat{b} - \underline{b})f(\hat{b})} = \frac{2}{\hat{b} - \underline{b}}.$$

Hence, $f(\hat{b}) - \Big(\frac{1}{\frac{a_L}{z}+\hat{b}} + \frac{1}{a_H+\hat{b}}\Big)F(\hat{b}) > 0$. This implies that if $a_H - \frac{a_L}{z} - \delta$ is large enough, it must be that $\frac{\partial q_0}{\partial b_1}|_{\hat{b}} > 0$. ■

Proof of Proposition 3. The price of new bonds is:

$$q_0 = \frac{1 - F(\tilde{b}_1 + b_1)}{\tilde{b}_1 - d_1 + b_1} + \frac{zF(\tilde{b}_1 + b_1)}{z\tilde{b}_1 - d_1 + zb_1}.$$

Taking the derivative, I obtain:

$$\frac{\partial q_0}{\partial b_1} = \frac{\partial q_0}{\partial b_1} = \frac{1}{\tilde{b}_1 + b_1 - d_1} \times$$

$$\times \Big\{\frac{2\frac{1-z}{z}d_1}{\tilde{b}_1 + b_1 - \frac{d_1}{z}}\Big(\frac{\tilde{b}_1 + b_1 - \underline{b}}{\bar{b} - \underline{b}}\Big)^2 \Big\{\frac{2}{\tilde{b}_1 + b_1 - \underline{b}} - \Big[\frac{1}{\tilde{b}_1 + b_1 - \frac{d_1}{z}} + \frac{1}{\tilde{b}_1 + b_1 - d_1}\Big]\Big\} - \frac{1}{\tilde{b}_1 + b_1 - d_1}\Big\}.$$

Define:

$$\hat{\lambda}(\tilde{b}_1+b_1)\equiv 2\frac{1-z}{z}d_1\Big(\frac{\tilde{b}_1+b_1-\underline{b}}{\overline{b}-\underline{b}}\Big)^2\Big\{\frac{2}{\tilde{b}_1+b_1-\underline{b}}-\Big[\frac{1}{\tilde{b}_1+b_1-\frac{d_1}{z}}+\frac{1}{\tilde{b}_1+b_1-d_1}\Big]\Big\}\,.$$

It suffices to show that $\hat{\lambda}(\frac{\underline{b}+\overline{b}}{2})\geq 1$. To see this, note that:

$$\hat{\lambda}(\frac{\underline{b}+\overline{b}}{2})=\frac{1-z}{z}d_1\Big\{\frac{2}{\overline{b}-\underline{b}}-\Big[\frac{1}{\underline{b}+\overline{b}-\frac{2d_1}{z}}+\frac{1}{\underline{b}+\overline{b}-2d_1}\Big]\Big\}>\frac{1-z}{z}d_1\Big\{\frac{1}{\overline{b}-\underline{b}}-\frac{1}{\underline{b}+(1-2z)\overline{b}}\Big\}\geq 1\,.$$

where I have used the fact that $d_1\leq z\underline{b}$. For the last part of the proof and to simplify the expositions, define $x\equiv\tilde{b}_1+b_1$. Taking the derivative of $\kappa\equiv(\tilde{b}_1+b_1-d_1)\frac{\partial q_0}{\partial b_1}$ with respect to d_1, I obtain:

$$\begin{aligned}\frac{\partial\kappa}{\partial d_1}=&2\Big(\frac{x-\underline{b}}{\overline{b}-\underline{b}}\Big)^2\Big\{\frac{(1-z)zx}{(zx-d_1)^2}\Big[\frac{2}{x-\underline{b}}-\Big(\frac{z}{zx-d_1}+\frac{1}{x-d_1}\Big)\Big]-\frac{(1-z)d_1}{zx-d_1}\Big[\frac{z}{(zx-d_1)^2}+\frac{1}{(x-d_1)^2}\Big]\Big\}-\\&-\frac{1}{(x-d_1)^2}\,.\end{aligned}$$

Therefore:

$$\begin{aligned}\frac{\partial\kappa}{\partial d_1}&>2\Big(\frac{x-\underline{b}}{\overline{b}-\underline{b}}\Big)^2\frac{(1-z)zx}{(zx-d_1)^2}\Big[\frac{2}{x-\underline{b}}-\frac{2z}{zx-d_1}\Big]-\frac{z}{(zx-d_1)^2}\Big\{\Big[2\Big(\frac{x-\underline{b}}{\overline{b}-\underline{b}}\Big)^2\frac{2(1-z)d_1}{zx-d_1}\Big]+1\Big\}=\\&=\frac{z}{(zx-d_1)^2}\Big\{4(1-z)\Big(\frac{x-\underline{b}}{\overline{b}-\underline{b}}\Big)^2\Big[\frac{x}{x-\underline{b}}-\frac{zx+d_1}{zx-d_1}\Big]-1\Big\}\,.\end{aligned}$$

For $\frac{\partial\kappa}{\partial d_1}>0$, it is sufficient that the term in braces in the last line above is positive. One has:

$$4(1-z)\Big(\frac{x-\underline{b}}{\overline{b}-\underline{b}}\Big)^2\Big[\frac{x}{x-\underline{b}}-\frac{zx+d_1}{zx-d_1}\Big]-1=\frac{4(1-z)}{(\overline{b}-\underline{b})^2}\frac{(x-\underline{b})(\frac{d_1}{z}\underline{b}+(\underline{b}-2\frac{d_1}{z})x)}{x-\frac{d_1}{z}}-1\,.$$

At $x = \frac{\underline{b}+\overline{b}}{2}$, the expression above, excluding the minus one at the end, becomes:

$$\frac{4(1-z)}{(\overline{b}-\underline{b})^2}\frac{(x-\underline{b})(\frac{d_1}{z}\underline{b}+(\underline{b}-2\frac{d_1}{z})x)}{x-\frac{d_1}{z}} = \frac{2(1-z)}{1-\frac{\underline{b}}{\overline{b}}}\frac{\frac{\underline{b}}{\overline{b}}\frac{\underline{b}+\overline{b}}{2}-\frac{d_1}{z}}{\frac{\underline{b}+\overline{b}}{2}-\frac{d_1}{z}} > 2(1-z)\left(\frac{\frac{\underline{b}}{\overline{b}}}{1-\frac{\underline{b}}{\overline{b}}}\right)^2 >$$

$$> 2(1-z)^2\left(\frac{\frac{\underline{b}}{\overline{b}}}{1-\frac{\underline{b}}{\overline{b}}}\right)^2\frac{\overline{b}-\underline{b}}{\overline{b}+(1-2z)\underline{b}} = (1-z)\underline{b}\Big\{\frac{1}{\overline{b}-\underline{b}} - \frac{1}{\underline{b}+(1-2z)\overline{b}}\Big\} \geq$$

$$\geq \frac{1-z}{z}d_1\Big\{\frac{1}{\overline{b}-\underline{b}} - \frac{1}{\underline{b}+(1-2z)\overline{b}}\Big\} \geq 1\,.$$

Hence, $\frac{\partial\kappa}{\partial d_1} > 0$ and consequently $\frac{\partial^2 q_0}{\partial d_1 \partial b_1} > 0$ at $x = \frac{\underline{b}+\overline{b}}{2}$. This impels that within a neighborhood of $\frac{\underline{b}+\overline{b}}{2}$, or for high enough values of $\tilde{b}_1 + b_1$, one must have $\frac{\partial^2 q_0}{\partial d_1 \partial b_1} > 0$. ■